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Global Existence of Solutions for a Model Boltzmann Equation

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A model recently introduced by Ianiro and Lebowitz is shown to have a global solution for initial data having a finite H-functional and belonging to $L_v^1(L_x^\infty)$. Methods previously introduced by Tartar to deal with discrete velocity models are used.

KEY WORDS: Kinetic theory; Boltzmann equation; hyperbolic equations.

1. INTRODUCTION

In a recent paper, Ianiro and Lebowitz[1] introduced a model kinetic equation describing the behavior of a gas between two walls. The model is essentially one-dimensional and does not conserve momentum. In fact, in order to avoid the trivial nature of collisions in a truly one-dimensional Boltzmann equation, fictitious collisions that reverse the velocities of particles traveling in opposite directions were introduced. The main justification of the model appeared to be its explicit solvability in the steady case with boundary conditions of the Maxwell type (perfect diffusion according to a Maxwellian at each wall).

In this paper we show that the model has another interesting feature: one can prove an existence theorem for the spatially inhomogeneous case and data of arbitrary size. The main tool is the extension of some results of Tartar[2,3] concerning a discrete velocity model to the Ianiro–Lebowitz (I(L) model.

In order to avoid unnecessary complications, in this paper we assume that only particles with bounded speed are present and move on an unbounded axis. It would be nice, of course, to extend these results to

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bounded intervals and in particular to investigate how the steady solution of Ianiro and Lebowitz is reached asymptotically in time, but this extension is not immediate.

2. THE MODEL

We consider a one-dimensional system of point particles moving on an infinite axis. Following Ianiro and Lebowitz,[1] we assume that the particles undergo collisions of the usual kind (i.e., collisions preserving the total momentum and energy of the particles), which have no effect on the distribution function, because they lead to an exchange of velocities between the particles, plus collisions that reverse the sign of the velocity of each particle. These are assumed to occur with probability p among particles traveling in opposite directions. If the distribution function $f = f(x, v, t)$ is normalized as a number density in phase space, the evolution equation to be solved is

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = p\int_{-\infty}^{\infty} |v - v_*|\, H(-vv_*)[f(x, -v, t)\, f(x, -v_*, t)$$
$$-f(x, v_*, t)\, f(x, v, t)]\, dv_* \tag{2.1}$$

with the initial condition

$$f(x, v, 0) = \phi(x, v) \tag{2.2}$$

with the condition

$$f \in L^1 \tag{2.3}$$

In Eq. (2.1) H denotes the usual Heaviside step function.

In order to simplify the subsequent treatment, we assume that

$$\phi(x, v) = 0 \qquad \text{if} \quad |v| > c \tag{2.4}$$

where c is a given positive constant, and look for a solution with the same property. A glance at the left-hand side of Eq. (2.1) indicates that this assumption is consistent, because no particle with speed greater than c can arise from collisions between particles with the same property. In other words, we solve the equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = p\int_{-c}^{c} |v - v_*|\, H(-vv_*)[f(x, -v, t)\, f(x, -v_*, t)$$
$$-f(x, v_*, t)\, f(x, v, t)]\, dv_* \equiv pQ(f, f)$$
$$-\infty < x < \infty; \qquad |v| \leqslant c \tag{2.5}$$

with the auxiliary condition (2.3) and the initial condition (2.2), where the initial data satisfy Eq. (2.4).

The constant p will be omitted in the following, since it can be removed by scaling.

3. PRELIMINARY RESULTS

In order to deal with Eq. (2.5), we first study the solutions of the simple equation

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}=Q(x, v, t), \qquad 0\leqslant t\leqslant T \tag{3.1}$$

with the initial and auxiliary cnditions (2.2) and (2.3). Here $Q(x, v, t)$ is a given function.

We denote by I an interval on the real axis and define

$$\begin{aligned} D=\{(x, t);\, t\in[0, T], &\text{ and there is a } v\in[-c, c]\\ &\text{and a } t\in[0,\ T] \text{ such that } x-vt\in I\}\end{aligned}$$

In problem, (2.1) v is just a parameter and this can be emphasized by writing $f_v(x, t)$ rather than $f(x, v, t)$.

The solution of the above problem is explicitly found to be

$$f_v(x, t)=\phi(x-vt)+\int_0^t Q_v(x-vs, t-s)\, ds \tag{3.2}$$

Here we can apply some simple results obtained by Tartar[3] in the case of discrete models. The fact that v will be taken to be variable in the interval $I_c=[-c, c]$ rather than taking discrete values will not play any role here.

Let us define the following function space:

$$\begin{aligned} F_v=(f_v(x, t) &\text{ defined on } D \text{ satisfying Eq. (3.2)}\\ &\text{with } Q_v\in L^1(D) \text{ and } \phi_v\in L^1(I))\end{aligned} \tag{3.3}$$

with norm

$$\|f_v\|_{V_v}=\|Q_v\|_{L^1(D)}+\|\phi_v\|_{L^1(I)} \tag{3.4}$$

F_v is a Banach space isometric to $L_1(D)\times L^1(I)$ [functions in F_v are $L^1_{\text{loc}}(D)$ and are only defined almost everywhere].

The following is a rather simple but useful result.

Lemma 1. If $v \in I_c$ and $f_v \in F_v$, then there is a $g_v \in L^1(I)$ such that

$$\begin{aligned} &|f_v(x, t)| \leqslant g_v(x-vt) \qquad \text{a.e. in } D \\ &\|g_v\|_{L^1(I)} = \|f_v\|_{F_v} \end{aligned} \tag{3.5}$$

Proof. By Eq. (3.2)

$$|f_v(x, t)| \leqslant |\phi_v(x-vt)| + \int_0^T |Q(x-vt+vs, s)|\, ds \qquad (0 \leqslant t \leqslant T) \tag{3.6}$$

Define

$$g_v(x) = |\phi_v(x)| + \int_0^T |Q(x+vs,\ s)|\, ds \qquad (x \in I) \tag{3.7}$$

Then the inequality in (3.5) is nothing else than (2.6) and the equality follows trivially from Eq. (3.4).

The above result produces an estimate of the L^1 norm of the product of two functions f_v and f_w $(v \neq w)$. This is the basic result to be used in the proof of an existence theorem in L^1 for small data:

Lemma 2. If $f_v \in F_v$ and $f_w \in F_w$ $(w \neq v)$, then $f_v f_w \in L^1(D)$ and

$$\|f_v f_w\|_{L^1(D)} \leqslant \frac{1}{|v-w|} \|f_v\|_{F_v} \|f_w\|_{F_w} \tag{3.8}$$

Proof. By Lemma 1 it is enough to bound

$$\int_D g_v(x-vt)\, g_w(x-wt)\, dx\, dt = \frac{1}{|v-w|} \int_{D'} g_v(y)\, g_w(z)\, dy\, dz \tag{3.9}$$

where D' is contained in $I \times I$. Equation (3.8) immediately follows thanks to Lemma 1.

4. GLOBAL EXISTENCE FOR SMALL L^1 DATA

We are now ready to prove the following result.

Theorem 4.1. There is a constant C_0 such that if $\phi_v \in L^1 \equiv L^1(R \times I_c)$ and satisfies

$$C \equiv |||\phi||| = \int_{-c}^{c} \|\phi_v\|_{L^1(R)}\, dv \leqslant C_0 \tag{4.1}$$

then there exist a unique solution f of Eq. (2.5). This solution satisfies

$$\int_{-c}^{c} \left\| \frac{\partial f_v}{\partial t} + v \frac{\partial f_v}{\partial x} \right\|_{L^1(R \times I_T)} dv \leqslant \int_{-c}^{c} \|\phi_v\|_{L^1(R)}\, dv = C \tag{4.2}$$

In order to prove this result, we use the contraction mapping theorem. To this end, we construct a mapping from $F = L^1(F_v)$ into F in the following way: Given $\phi \in L^1$ and $h \in F$, $f = N(h)$ is the solution of

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = Q(h, h), \qquad f(x, v, t) = \phi(x, v) \tag{4.3}$$

where $Q(h, h)$ is defined in Eq. (2.5).

Proof. By Lemma 2 of the previous section we have

$$\left\| \int Q(h, h)\, dv \right\|_{L^1(D)} \leqslant 2 \|h\|_F^2 \tag{4.4}$$

i.e., the right-hand side of Eq. (4.3) is in L^1. It is clear now that a mapping $h \to f = N(h)$ is established. We want to see that this mapping is a contraction on some closed set of F.

In fact, Eqs. (3.4), (4.3), and (4.4) give

$$\|f\|_F \leqslant \|\phi\|_{L^1} + \|Q(h, h)\|_{L^1(D)} \leqslant \|\phi\| + 2 \|h\|_F^2 \tag{4.5}$$

Then N maps the ball B_R into another ball $B_{\hat{R}}$ if

$$\hat{R} \geqslant C + 2R^2 \tag{4.6}$$

We can now bound the Lipschitz constant of N in B_R. If $\bar{h}$ is another function in this ball and $\bar{f} = N(\bar{h})$, we have

$$\|f - \bar{f}\|_F \leqslant 4R \|h - \bar{h}\| \tag{4.7}$$

Now if we define C_0 to be 3/32 and take $R = 1/8$, we have a strict contraction in a ball and the theorem is proved. In particular, Eq. (4.2) follows from (4.5) with $h = f$.

We can now prove a simple result on the continuous dependence on the initial data.

Theorem 4.2. If ϕ, $\bar{\phi} \in L^1$ and their norms are less than C_0, the corresponding solutions f and $\bar{f}$ satisfy

$$\sup_{t \in R} \|f - \bar{f}\|_{L^1} \leqslant 2 \|\phi - \bar{\phi}\|_{L^1} \tag{4.8}$$

It is sufficient to modify (4.7) in order to take into account the difference in initial data. Then

$$\|f-\bar{f}\|_F \leqslant \|\phi-\bar{\phi}\|_{L^1} + \tfrac{1}{2}\|f-\bar{f}\|_F \tag{4.9}$$

and (4.8) follows, given the definition of the norm in V.

5. LOCAL EXISTENCE FOR ARBITRARY L^1 DATA AND L^∞ REGULARITY FOR SMALL L^1 DATA. EXISTENCE FOR DATA OF ARBITRARY SIZE

The theorem given in the previous section is restricted to small data in L^1. In order to move to arbitrarily large data, we first show a uniqueness result for solutions not necessarily lying in B_R.

Theorem 5.1. If f and $\bar{f}$ are solutions taking the same data ϕ at $t=0$, then f and $\bar{f}$ coincide in $[-t_0, t_0]$ for some $t_0>0$.

To prove this, we note that since the left-hand side of (4.3) is in L^1, we can take a domain $S=J\times I_t$ (where $J=[x_0, x_0+r]$ and $I_t=[-t_0, t_0]$) with measure less than δ, in such a way that if we put the initial data equal to zero outside J, then their norm in L^1 will be less than C_0 and the associated solutions will have norm smaller than R; thus, they coincide in S because of Theorem 4.1. But because of hyperbolicity, they coincide with f and $\bar{f}$ in the domain $\hat{S}$, where

$$\hat{S} = \{(x, t) \in S \text{ with } x \pm ct \text{ in } J\}$$

Hence f and $\bar{f}$ coincide in $\hat{S}$. By moving x_0 in R, they also coincide in $R\times I_t$.

We can now prove the following result.

Theorem 5.2. (Local existence for L^1 data.) Let $\phi \in L^1$; then there exists a $t_0>0$ such that Eq. (4.3) has a solution on the interval $[-t_0, t_0]$.

As in the previous theorem, we exploit hyperbolicity. In fact, we can find a finite number of intervals J_k $(k=1,..., q)$ such that their union gives R and such that the restriction of ϕ to each interval has L^1 norm less than C_0. If we take initial data equal to ϕ in each J_k and zero outside, we find solutions in a corresponding domain $\hat{S}_k$, to be denoted by f^k. If J_k and J_h have a nonvoid intersection, then f^k and f^h must coincide on the intersection of $\hat{S}_k$ and $\hat{S}_h$. Since the union of all $\hat{S}$'s contains a strip $[-t_0, t_0]$ (thanks to $|v|\leqslant c$), we can glue the different f's together to obtain a solution in $[-t_0, t_0]$ and the theorem is proved.

It is now possible to prove an L^∞ result in the following form.

Theorem 5.3. There exists $C_1 > 0$, $k > 1$, such that, if ϕ is in the intersection of $L_v^1(L_x^\infty)$ and $L_{x,v}^1$ with L^1 norm less than C_1, then the solution is essentially bounded in $I_L \times R_t$ a.e. in v and satisfies

$$\int \|f\|_{L^\infty(I_L \times R_t)} \leqslant k \int \|\phi\|_{L^\infty(I_L)}\, dv \tag{5.1}$$

Let us define $M(t)$ to be the integral with respect to v appearing on the left-hand side; then $M(0)$ will be the integral with respect to v appearing on the right-hand side. We want to bound $M(t)$ in terms of $M(0)$.

We know that f satisfies Eq. (4.3). Accordingly, we bound the integral

$$S = \int_0^t |v - v_*| \, [f(x - vs, -v, t - s) f(x - vs, -v_*, t - s) + f(x - vs, v, t - s) f(x - vs, v_*, t - s)] \, dv_* \, ds \tag{5.2}$$

If we take into account the elementary inequality $|v - v_*| \leqslant |v + v_*| + 2|v|$ and Lemma 1, we immediately find

$$S \leqslant \|f(\cdot, -v, \cdot\|_{L^\infty} \|f\|_F + \left[\int g(x, -v)\, dx\right] M(t) + \|f\|_F \|f(\cdot, v, \cdot)\|_{L^\infty} \tag{5.3}$$

Then a bound for $f(x, v, t)$ immediately follows and by integrating with respect to v after taking the supremum with respect to x and t, we obtain

$$M(t) \leqslant M(0) + 3M(t)\, \|f\|_F \tag{5.4}$$

But $\|f\|_F \leqslant 2C_1$ [thanks to Eq. (4.2)] and hence if $C_1 = 1/6$, we have

$$M(t) \leqslant \frac{M(0)}{1 - 6C_1} \tag{5.5}$$

This estimate is valid as long as the solution is bounded. Since, however, the bound is independent of t, we have a solution in $L^1(L^\infty)$ for any t.

We are ready now to prove global existence in $L^1(L^\infty)$ for data of any size:

Theorem 5.4. If the data have a finite H-functional H_0, then the solution exists in $L^1(L^\infty)$ without restrictions on the size of the data (assumed to be in the intersection of $L_x^1(L_v^\infty)$ and L^1).

By standard arguments we prove that the solution of the previous theorems is nonnegative for $0 \leqslant t \leqslant T$. In order to prove the theorem we need to find a bound on the norm of f in $L^1(L^\infty)$ depending only upon T and the data. Let $I=[x_0-cT,\ x_0+cT]$ and define ψ to be

$$\psi=\begin{cases}\phi & \text{if}\quad x\in I\\ 0 & \text{if}\quad x\notin I\end{cases}\tag{5.6}$$

The solution h with initial data ψ will coincide with f at (x_0, T) and will have compact support for each time t between 0 and T. In particular, there will be constant k such that the measure of these supports will be less than kT. Using the H-theorem for h, we have

$$\int h\log h\,dx\,dv\leqslant H_0\tag{5.7}$$

(We remark that the rigorous proof of the H-theorem requires tedious but well known steps; see, e.g., Ref. 4. Hence

$$\int h\log^+ h\,dx\,dv\leqslant H_0+2ce^{-1}kT\equiv C(H_0, T)\tag{5.8}$$

We now show that we can choose an r depending only upon H_0 and T such that

$$\begin{aligned}I&=\iint_x^{x+r} h(z, v, t)\,dz\,dv\\ &\leqslant C_1 \text{ uniformly for } x\in R,\ t\in[0, T]\end{aligned}\tag{5.9}$$

To this end we decompose the integral I into two parts, I_1 and I_2; I_1 is over the set where $h\geqslant N$, where N is for the moment arbitrary but larger than unity, and I_2 over the set $h\leqslant N$ (if the set where $h=N$ is of nonzero measure, I_1+I_2 is larger than I, but this does not matter, since the inequality is in the right direction). Then

$$I_1\leqslant\frac{1}{\log N}\iint_x^{x+r}\log^+ h\,h\,dz\,dv\leqslant\frac{C(H_0, T)}{\log N}\tag{5.10}$$

$$I_2\leqslant 2cNr\tag{5.11}$$

If is sufficient now to choose $N\geqslant\exp[2C(H_0, T)/C_1]$ and

$$r\leqslant\frac{C_1}{4c}\exp\left(-\frac{2C(H_0, T)}{c_1}\right)\tag{5.12}$$

to have Eq. (5.9).

This means that we can apply Theorem 5.3 a finite number of times (to be precise, $2cT/r$ times) to obtain a bound for $M(t)$ for $0 \leqslant t \leqslant T$ and thus existence in $L_v^1(L_x^\infty)$.

6. CONCLUDING REMARKS

The result of global existence for the IL model proved in this paper indicates that the bounds required by global existence theorems can depend on tiny details of the collision model, not necessarily tied to its nonlinear structure. In fact, the extension of the present treatment to the actual Boltzmann equation, if possible, is far from trivial.

Some of the results of the present paper can be extended to a bounded domain with periodicity conditions; to this end the hyperbolic nature of the equation has to be exploited in a suitable fashion.

REFERENCES

1. N. Ianiro and J. L. Lebowitz, *Found. Phys.* **15**:531 (1985).
2. L. C. Tartar, Séminaire Goulaouic-Schwarz (1975/76), No. 1.
3. L. C. Tartar, University of Wisconsin–Madison, Mathematics Research Center (1980).
4. L. Arkeryd, *Arch. Rat. Mech. Anal.* **45**:1 (1972).